Subject: ON THE EFFECT OF LONGITUDINAL SPACE CHARGE FORCES ON A DRIFTING BUNCHED BEAM

Introduction

If a bunched beam is allowed to drift for some distance, the self-force will speed up the leading particles and retard the lagging ones. To the extent that the force is linear in displacement of a particle from the center of gravity of the bunch, this effect is not too serious, since the effective area occupied by the particles in phase space remains constant and the mis-orientation of the enclosing ellipse can be corrected by a suitable adjustment of the externally applied forces. However, the self-force is certainly non-linear to some degree, both in dependence on longitudinal position and on amplitude of transverse motion; the resulting distortion and smearing in phase space can not be undone in practice.

Since the non-linear contributions from a reasonably smooth density distribution within a bunch tend to be smaller than the linear contribution, and since a formalism exists for handling the linear problem exactly, an estimate of the effect can be obtained by linearizing the forces. If the resulting numbers are negligible, the effect is probably negligible; if not, a more high powered treatment may be indicated. The numerical examples in this note are for the case of a linac beam drifting between tanks or from final tank to a debuncher and beyond, the situations which prompted this investigation.

The Envelope Equation

If the initial distribution of particles in longitudinal phase space is contained within a bounding ellipse and if the self-forces are linear in longitudinal displacement, then in the course of time the boundary will remain elliptical, of constant area but changing orientation and axis ratio. At any moment, it can be characterized by three parameters, as follows:

$$W^2 = z_m^{\,2}(t)p^2 + p_m^{\,2}(t)z^2 - 2\sqrt{z_m^{\,2}(t)p_m^{\,2}(t) - W^2}\,pz, \qquad (1)$$

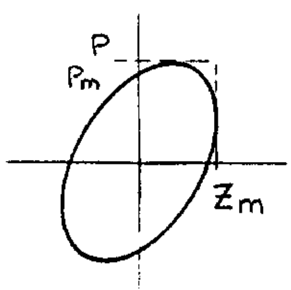
where p and z are, respectively, deviations in longitudinal momentum and position from the average and

$W = \frac{1}{\pi}$ (ellipse area)

$z_m(t)$ = maximum value of z on the ellipse

$p_m(t)$ = maximum value of p on the ellipse

as indicated in figure 1.



z and p obey the equations of motion:

$$\frac{dz}{dt} = \frac{p}{M\gamma^3} \qquad (2)$$

$$\frac{dp}{dt} = kz \tag{3}$$

where M is the particle rest mass, γM the total energy of reference particle, and k is the linearized coefficient representing the space charge force, the explicit form of which will be discussed in the next section.

Equations of motion for z_m and p_m can be obtained by differentiating Eq. (1) and using relations (2) and (3). The result is

$$\frac{dz_m}{dt} = \frac{1}{M\gamma^3 z_m} \sqrt{p_m^2 z_m^2 - W^2} \tag{4}$$

$$\frac{dp_m}{dt} = \frac{k}{p_m} \sqrt{p_m^2 z_m^2 - W^2} \tag{5}$$

This pair of equations is entirely equivalent to (2) and (3), but much better suited to the purpose since they refer directly to the interesting features of the bounding ellipse. It should be noted that they are quite non-linear, particularly since k can be a complicated function of z_m, according to the model used to estimate the space charge force. Thus (2) and (3) are not directly soluble, since k is an undetermined function of time.

By differentiating Eq. (4) and substituting (5) where needed, we arrive at an equation for the length of the bunch as a function of time:

$$\frac{d^2}{dt^2} z_m = \frac{k(z_m)}{M\gamma^3} z_m + \frac{W^2}{(M\gamma^3)^2 z_m^3} , \tag{6}$$

which is known as the envelope equation.

For the purpose of seeing how much p_m is affected by the self-force, it is convenient to obtain a first integral directly by dividing (5) by (4) and integrating. This yields:

$$p_m^2 - p_{mi}^2 = 2M\gamma^3 \int_{z_{mi}}^{z_m} k(z_m) z_m dz_m,$$

where the subscript denotes initial values. For any reasonable model of the space charge force, $k(z_m)$ decreases sufficiently rapidly for large z_m that the integral is finite, even for a single bunch drifting forever. Thus, the magnitude of the effect for a beam traveling to a debuncher is given by:

$$\Delta p_m^2 \leq 2M\gamma^3 \int_{z_{mi}}^{\infty} k(z_m) z_m dz_m \tag{8}$$

<u>Determination of $k(z_m)$</u>

As a model for the distribution of particles in real space we take a cylinder of length $2z_m$ and radius, a, moving down the axis of a conducting pipe of radius, b. The density is assumed to be uniform in radius but decreasing parabolically to zero at the ends of the bunch; i.e., $\rho \sim 1 - \dfrac{z^2}{z_m^2}$.

The induced charges on the pipe are assumed to follow the same dependence on z, which model is probably accurate enough for the range of parameters in question and certainly simpler to handle than the exact Green's function expressions. Finally, the longitudinal field on the axis is assumed to act on all particles, regardless of their transverse positions.

This model is about the most realistic that can be handled analytically. It can be seen from the formulas below that the field on axis increases for large z more slowly than linearly, so that an overestimate of the effect should follow from using a value of k corresponding to small z. It should be noted, however, that such properties are quite model-sensitive; for instance, in a uniformly charged cylinder, the force constant at the center is smaller, but the field increases rapidly toward the ends of the bunch.

It can be shown by applying Hamiltonian transformations to the phase-space ellipse of Fig. 1 that, to the extent that the transformations are linear, the assumed form of the density distribution in z is maintained during the motion, the only change being in the time-dependent scale factor, z_m. This point of consistency is essential to this method of calculation.

The full expression for the field on axis is[1]

$$eE(z) = \frac{3Ne^2}{a^2 z_m^3} \left\{ z z_m^2 + \frac{za^2}{2\gamma^2} \left[\sinh^{-1} \gamma \frac{z_m + z}{a} \right.\right.$$

$$+ \sinh^{-1} \gamma \frac{z_m - z}{a} + \gamma^2 \frac{z_m + z}{a^2} \sqrt{(z_m + z)^2 + \frac{a^2}{\gamma^2}}$$

$$\left. + \gamma^2 \frac{z_m - z}{a^2} \sqrt{(z_m - z)^2 + \frac{a^2}{\gamma^2}} \right] - \frac{1}{3} z^3$$

$$\left. + \frac{1}{3}\left[\left((z_m - z)^2 + \frac{a^2}{\gamma^2} \right)^{3/2} - \left((z_m + z)^2 + \frac{a^2}{\gamma^2} \right)^{3/2} \right] \right\}$$

$$- \frac{3}{2} \frac{Ne^2}{\gamma^2 z_m^3} \left\{ z \left[\sinh^{-1} \gamma \frac{z_m + z}{b} + \sinh^{-1} \gamma \frac{z_m - z}{b} \right] \right.$$

$$\left. + \sqrt{(z_m - z)^2 + \frac{b^2}{\gamma^2}} - \sqrt{(z_m + z)^2 + \frac{b^2}{\gamma^2}} \right\}, \tag{9}$$

where N is the total number of particles in the bunch.

Retaining linear terms in z:

$$eE(z) \sim \frac{3Ne^2}{a^2} \frac{z}{z_m} \left[1 + \frac{a^2}{\gamma^2 z_m^2} \sinh^{-1} \gamma \frac{z_m}{a} - \sqrt{1 + \frac{a^2}{\gamma^2 z_m^2}} \right]$$

$$- \frac{3Ne^2}{\gamma^2} \frac{z}{z_m^3} \left[\sinh^{-1} \gamma \frac{z_m}{b} - \left(1 + \frac{b^2}{\gamma^2 z_m^2} \right)^{-1/2} \right] \tag{10}$$

so that

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[1] The fact that the bunch is in motion in the lab system is taken into account by using the elementary expression for the force between two moving particles; see, for example, Smythe, sect. 16.05, Eq. (2).

$$z_m k(z_m) = \frac{3Ne^2}{a^2}\left[1 + \frac{a^2}{\gamma^2 z_m^2}\ \sinh^{-1}\ \gamma\frac{z_m}{a} - \sqrt{1 + \frac{a^2}{\gamma^2 z_m^2}}\right]$$

$$- \frac{3Ne^2}{\gamma^2 z_m^2}\left[\sinh^{-1}\ \gamma\frac{z_m}{b} - \left(1 + \frac{b^2}{\gamma^2 z_m^2}\right)^{-1/2}\right]$$

$$= \frac{3Ne^2}{a^2}\left[1 - \frac{a}{\gamma z_m}\sqrt{1 + \frac{\gamma^2 z_m^2}{a^2}} + \frac{a^2}{b^2}\left(1 + \frac{\gamma^2 z_m^2}{b^2}\right)^{-1/2}\right.$$

$$\left. + \frac{a^2}{\gamma^2 z_m^2}\ \ell n\left(\frac{\frac{\gamma z_m}{a} + \sqrt{1 + \frac{\gamma^2 z_m^2}{a^2}}}{\frac{\gamma z_m}{b} + \sqrt{1 + \frac{\gamma^2 z_m^2}{b^2}}}\right)\right] . \qquad (11)$$

Application to Debunching

If the beam is allowed to drift for such a distance that the final spatial extent is large compared to the initial one, Eq. (7) is best suited to show the effect of the self-forces. After dividing by the square of the reference momentum and performing the indicated integration on relation (11), we obtain:

$$\left(\frac{\Delta p}{p}\right)_F^2 - \left(\frac{\Delta p}{p}\right)_i^2 = \qquad (12)$$

$$\frac{6Ne^2}{Mc^2\beta^2 a}\left\{\frac{a}{\gamma z_m}\ \sinh^{-1}\ \frac{\gamma z_m}{a} + \sqrt{1 + \frac{\gamma^2 z_m^2}{a^2}} - \frac{a}{\gamma z_m}\ \sinh^{-1}\ \frac{\gamma z_m}{b}\right\}_{Z_{mF}}^{Z_{mi}}$$

where $\frac{\Delta p}{p}$ has its usual meaning and the subscripts refer to initial and final values.

Before substituting specific numbers for the NAL linac, it might be worth pointing out a few surprising features of Eq. (12). The relation between increased momentum spread and beam current is quadratic rather than linear; i.e., if the increase is large, it is proportional to the square root of intensity. Furthermore, the area of the ellipse in phase space does not appear explicitly, nor does its initial orientation. Finally, the increase is not as sensitive to beam radius as was indicated by a more primitive calculation.

The right hand member of Eq. (12) is given in Table I for various values of a and z_m, assuming a beam current of 100 ma at 200 MeV and 200 Mc, and taking b to be 4 cm or (a+1) cm, which ever is larger. The quantities a and z_m are expressed in centimeters.

Table I

	$10^6 \Delta \left(\frac{\Delta p}{p}\right)^2$						
a\\z_m	0	1.	2.	4.	6.	8.	10.
1/2	25.7	11.3	7.2	4.1	2.8	2.2	1.8
1	12.0	7.2	4.8	2.9	2.1	1.6	1.3
4	2.1	1.6	1.4	0.96	0.75	0.62	0.48

The table can be used in the following way: if, for example, $\frac{\Delta p}{p} = 10^{-3}$, $z_{mi} = 1$ cm (4° at 200 Mc), $z_{mF} = 4$ cm (expansion in a drift distance of about 50 meters), and a = 1 cm,

$$\left(\frac{\Delta p}{p}\right)_F = \left[(10^{-3})^2 + 10^{-6}(7.2 - 2.9)\right]^{1/2} = 2.3\times10^{-3}$$

or, for a = 4 cm,

$$\left(\frac{\Delta p}{p}\right)_F = \left[(10^{-3}) + 10^{-6}(1.6 - .96)\right]^{1/2} = 1.3\times10^{-3}$$

One might conclude that an expansion to a large radius is helpful but, pursuing the beam farther, its radius must soon be brought down to ~ 1 cm to match the booster requirements. Thus, even if the debuncher reduced the spread to a negligible value, the final momentum spread after complete debunching in the booster would be:

$$\left(\frac{\Delta p}{p}\right)_F = \left[0 + 10^{-6}(2.9 - 0)\right]^{1/2} = 1.7\times10^{-3}$$

Another interesting example is z_{mi} = 2 cm (3° at 200 Mc), a = 1 cm, $\left(\frac{\Delta p}{p}\right)_i = 10^{-3}$, and no debuncher cavity. Then

$$\left(\frac{\Delta p}{p}\right)_F = \left[(10^{-3})^2 + 10^{-6}(4.8 - 0)\right]^{1/2} = 2.4\times10^{-3},$$

which would indicate that one might do as well by manipulating the beam in the linac as by adding a separate cavity. The separate cavity should be very useful at low beam current, but the present analysis suggests that in the 100 ma range it would only serve to make matters not quite as bad as they might otherwise be.

Application to Inter-Tank Drift Spaces

In passing from one tank to the next, the bunch shape in phase space is distorted even at low intensity, which makes it difficult to establish a criterion for the seriousness of space-charge effects. We shall assume that the low intensity distortion is not damaging, in the sense that it gives rise to a mismatch but the effective phase-space area is conserved. The procedure will be to compute the additional distortion due to space charge, find the area of an ellipse similar to the low intensity one which encloses the actual ellipse, and use the fractional increase in that area as a measure of the space-charge effect. The tacit assumption is that the present formalism, though linear, indicates the magnitude of the non-linear contributions, so that the change in area represents a real loss in quality. Incidentally, this assumption is not involved in the de-bunching case because in the process of de-bunching and re-bunching in the booster the inherent quality is lost and only the total momentum spread is significant, irregardless of whether it arose from linear or non-linear effects.

The ratio of the area of a "matched" (in the sense defined above) low intensity ellipse to the enclosed ellipse resulting from the linearized treatment of space charge effects can be shown to be:

$$W_{sc}/W = A + \sqrt{A^2 - 1} \tag{14}$$

where $$A = \frac{1}{2W^2}\left\{ p_{sc}^2 z_d^2 + p_d^2 z_{sc}^2 - 2\left[\left(p_d^2 z_d^2 - W^2\right)\left(p_{sc}^2 z_{sc}^2 - W^2\right)\right]^{1/2}\right\}, \qquad (15)$$

in which p_d and z_d are the values of p_m and z_m after a drift at zero intensity and p_{sc} and z_{sc} are the corresponding quantities in the presence of space charge.

These four quantities can be obtained for arbitrary size and orientation of the ellipse at the end of a tank by integrating Eq. (6) through the drift space, first with k = 0 and then with a value of k corresponding to the actual beam intensity. This procedure determines z_m; Eq. (7) then gives p_m. The present report will be restricted to an estimate of the space charge effect in the approximation that the beam leaving a tank is represented by an ellipse lined up with the axis system ($p_m z_m = W$) and that the change is z_m in traversing the drift space is sufficiently small that z_m may be considered constant in Eq. (6). In this case, Eq. (14) boils down to:

$$\frac{W_{sc} - W}{W} = \frac{Lk(z_m)z_m}{\Delta E} \qquad (16)$$

where L = length of drift space

ΔE = half width of energy spread

Table II illustrates the magnitude of the effect for the NAL linac according to Eq. (16), assuming a current of 100 ma, a = 1 cm, b = 4 cm, an injection phase spread of $\pm 30^\circ$

and energy spread of ±50 keV, and assuming normal damping to obtain a phase and energy spread at the end of each tank. It might be tempting to obtain the overall effect by multiplying the eight $\frac{W_{sc}}{W}$ ratios (~2.2), but this would be misleading since the beam is generally mismatched for various reasons and the option of adjusting tank amplitudes and phases has been ignored.

Table II

Drift Space No.	Length (m)	z_m (cm)	$\frac{\Delta E}{L}$ $\frac{\text{keV}}{\text{cm}}$	$\frac{W_{sc}}{W}$
1	.22	.70	6.2	1.06
2	.60	.81	3.5	1.11
3	.75	.87	3.6	1.11
4	1.0	.89	2.9	1.13
5	1.0	.92	3.1	1.12
6	1.0	.93	3.3	1.11
7	1.0	.95	3.4	1.10
8	1.0	.96	3.5	1.10

Further Work

A computer program has been written by A. Warren to solve the basic equation (6). Although the first integral can be obtained analytically [Eq. (12)], it is inadequate to describe what happens in the real case of drifting for a given distance or time. It is planned to use this program to examine specific linac-booster configurations, and to investigate the tank-to-tank drift for a variety of possible beam configurations. An attempt will also be made to study pre-bunching phenomena; in this case, the initial beam configuration on leaving the buncher is far from the idealized ellipse assumed in this analysis, but it should be possible to get some idea of the influence of space-charge forces on the bunching process.